The correct basic equations for the one-dimensional channel flow of a partially ionized gas with heat addition are

$$(d/dx)(\rho A u) = 0 \tag{1}$$

$$\rho u(du/dx) + (dp/dx) = 0 \tag{2}$$

$$\rho u(d/dx)[h + (u^2/2)] = q \tag{3}$$

$$\frac{\alpha^2 p}{1-\alpha^2} = \frac{(2\pi m_e)^{3/2}k^{5/2}}{(2\pi\bar{h})^3}\,\frac{2(g_0)_i}{(g_0)_a}\,T^{5/2}\exp - \left(\frac{X}{kT}\right) \tag{4}$$

$$p = \rho(1+\alpha)(k/m)T \tag{5}$$

The specific enthalpy of a partially ionized monatomic gas is derived by Jones and McChesney[2] as

$$h = \tfrac{5}{2}(kT/m)(1+\alpha) + (\alpha X/m) \tag{6}$$

$$\left[\frac{\dfrac{nkT}{m}\dfrac{1+\alpha}{2-\alpha}\left\{\alpha(1-\alpha)\left(\dfrac{5}{2}+\dfrac{X}{kT}\right)^2+5\right\} - nu^2\left\{\dfrac{3}{2}(1+\alpha)+\dfrac{\alpha(1-\alpha)}{(2-\alpha)}\left(\dfrac{3}{2}+\dfrac{X}{kT}\right)^2\right\}}{(1+\alpha)+\dfrac{\alpha(1-\alpha)}{(2-\alpha)}\left(\dfrac{3}{2}+\dfrac{X}{kT}\right)}\right]\frac{du}{dx} + \left[\frac{\dfrac{nukT}{m}\dfrac{1+\alpha}{2-\alpha}\left\{\alpha(1-\alpha)\left(\dfrac{5}{2}+\dfrac{X}{kT}\right)^2+5\right\}}{(1+\alpha)+\dfrac{\alpha(1-\alpha)}{2-\alpha)}\left(\dfrac{3}{2}+\dfrac{X}{kT}\right)}\right]\frac{1}{A}\frac{dA}{dx} - \frac{q}{m} = 0 \tag{15}$$

This equation is valid provided electronic excitation of the atomic and ionic species can be ignored and there is no multiple ionization. This approximation has also been made in the Saha equation, Eq. (4), where the species statistical weights replace the partition functions.

Combining Eqs. (3) and (6) gives the energy equation

$$nu^2\frac{du}{dx} + \frac{5}{2}\frac{k}{m}nu\frac{d}{dx}[(1+\alpha)T] + \frac{nuX}{m}\frac{d\alpha}{dx} = \frac{q}{m} \tag{7}$$

The energy equation used by Yen is similar in appearance to (7) except that he writes the second term on the left-hand side as

$$\frac{\gamma}{\gamma-1}\frac{k}{m}nu\frac{d}{dx}[(1+\alpha)T] \tag{8}$$

Here γ is the ratio of the specific heats which Yen takes as $\frac{5}{3}$. Furthermore, Yen introduces the sound speed a in the partially ionized gas as a whole as

$$\begin{aligned} a^2 &= \tfrac{5}{3}(k/m)(1+\alpha)T \\ &= \gamma(k/m)(1+\alpha)T \end{aligned} \tag{9}$$

Equations (8) and (9), which are Yen's, are incorrect for a partially ionized gas. The reasons are discussed in detail by Jones and McChesney where it is shown that, for a reacting gas mixture, the specific heat ratio γ does not characterize isentropic processes in the flow as it does for an ideal gas. In the paper just cited, it is shown that the correct expression for the equilibrium sound speed is

$$a^2 = \gamma^*(k/m)(1+\alpha)T \tag{10}$$

where

$$\gamma^* = \frac{2\gamma}{(2-\alpha)(1+\alpha)} \tag{11}$$

$$\gamma = \frac{5(1+\alpha)+\alpha(1-\alpha^2)[\frac{5}{2}+(X/kT)]^2}{3(1+\alpha)+[2\alpha(1-\alpha)/(2-\alpha)][\frac{3}{2}+(X/kT)]^2} \tag{12}$$

It is possible to characterize a real gas flow energy equation with a term similar to Eq. (8). Namely, we can write Eq. (8) as

$$\frac{\gamma'}{\gamma'-1}\frac{k}{m}nu\frac{d}{dx}[(1+\alpha)T] \tag{13}$$

Here, $\gamma' = h/e$, where e is the specific internal energy, i.e.,

$$h = [\gamma'/(\gamma'-1)](p/\rho)$$

which is analogous to the perfect gas relation. With the approximation of Eqs. (4) and (6), our expression for γ' becomes

$$\gamma' = \frac{5(1+\alpha)+(2\alpha X/kT)}{3(1+\alpha)+(2\alpha X/kT)} \tag{14}$$

If we solve Eqs. (1–5), then after considerable algebraic reduction we obtain Eq. (15).

Choking occurs when the numerator of the coefficient of du/dx vanishes; this gives $u = a$, where a is given by the foregoing Eqs. (9–11), i.e., we get choking at Mach 1.

This result would have been expected from the studies of Shercliff,[3] who has shown that the speed of sound is a critical velocity for any single phase fluid in equilibrium provided certain criteria are obeyed (which, for a partially ionized gas, are obeyed).

Equation (15) reduces to the usual equation for perfect gas flow when $\alpha = 0$ and 1 where the usual relations between h, γ, p, ρ and a, γ, p, ρ hold. However, in the fully ionized case, the Saha equation loses its meaning.

References

[1] Yen K. T., "Thermal choking of partially ionized gases," AIAA J. **1**, 2171–2172 (1963).

[2] Jones, N. R. and McChesney, M., Equilibrium ionization calculations for normal and oblique shock wave in argon," Proc. Phys. Soc. (London) **81**, 1022–1033 (1963).

[3] Shercliff, J. A., "Some generalizations in steady one-dimensional gas dynamics," J. Fluid Mech. **3**, 645–657 (1958).

Reply by Author to M. McChesney

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In my note,[1] n is meant to be the original number density of the neutral particles (i.e., when $\alpha = 0$). Thus, $n_n = (1-\alpha)n$, $n_e = n_p = \alpha n$, and the total number of particles is $n_n + n_e + n_n = (1+\alpha)n$. The total pressure is $p = (1+\alpha)nkT$. Of course, $n = n_n + n_p$. The energy equation (7) given for monatomic gases in McChesney's comment is identically the same as the one given in Ref. 1, with γ taken to be $\frac{5}{3}$. Therefore, my basic equations are correct.

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In my note, it is stated that "Eq. (10) suggests the possibility of defining a Mach number, which becomes one at choking for $0 \leqslant \alpha \leqslant 1$." The corresponding speed of sound can be obtained from Eq. (10):

$$C_e = [\Gamma(5k/3m)(1 + \alpha)T]^{1/2} \tag{1}$$

where

$$\Gamma = \frac{1 + [(\gamma - 1)/\gamma](2.5 + t_i)F}{1 + (\gamma - 1)(1.5 + t_i)F}$$

with

$$F = \frac{t_i \alpha \chi}{2 + (2 + 1.5\chi + \chi t_i)} \qquad \chi = \frac{2(1 - \alpha)}{2 - \alpha}$$

By taking $\gamma = \frac{5}{3}$ for monatomic gases and by performing some algebraic manipulations, the expression Γ can be written in the following form:

$$\Gamma = \frac{[5/\alpha(1 - \alpha)] + (2.5 + t_i)^2}{\frac{3}{2}[(1 + \alpha)(2 - \alpha)/\alpha(1 - \alpha)] + (1.5 + t_i)^2} \tag{2}$$

This expression agrees with γ^* given by Eq. (12) obtained by Jones and McChesney.[2] Therefore, C_e given in Eq. (1) is indeed the equilibrium speed of sound. This fact also establishes the correctness of the basic equations used in my note.

Although in my note the correct equilibrium speed of sound was obtained, it was not identified as such. To be consistent, the equilibrium speed of sound should be used for equilibrium flows. This appears to be the point in McChesney's comments.†

The fact that equilibrium speed of sound comes out automatically from the basic equations and the condition of choking suggests the following point: the frozen speed of sound may be used even for equilibrium flows provided some care is used in interpreting the results. Finally, it is to be noted that, for nonequilibrium flows, a meaningful speed of sound cannot be defined such that choking will occur at Mach number equal to 1.

References

[1] Yen, K. T., "Thermal choking of partially ionized gases," AIAA J. 1, 2171–2172 (1963).

[2] Jones, N. R. and McChesney, M., "Normal and oblique shock waves in argon," Proc. Phys. Soc. (London) 81, 1022–1033 (1963).

† Choking at equilibrium Mach number 1 has been pointed out earlier to the author by David C. Dryburgh of Rolls-Royce Limited, England.

Transverse Curvature Effects in Axisymmetric Hypersonic Boundary Layers

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It has been shown that mass transfer affects the induced surface pressure resulting from shock-wave boundary-layer interaction on a flat plate.[1, 2] A logical extension of this problem is to broaden these considerations to include axisymmetric configurations that would require the retention of transverse curvature terms in the boundary-layer equations. The authors have considered this problem, and the results will be presented in Ref. 3. In matching the inviscid and viscous flow solutions, it is necessary to investigate the effects of surface blowing on the vertical velocity component at the boundary between the inviscid and viscous flow regions. In this note, the expression for v_e/u_e will be derived. This result is compared with the recently published formula of Thyson and Schurmann.[4]

Consider the continuity equation for a nonreacting gas flowing on the surface of an axisymmetric body defined by $r_w = r_w(x)$:

$$(\partial/\partial x)(\rho u r) + (\partial/\partial y)(\rho v r) = 0 \tag{1}$$

where $r = r_w + y \cos\alpha$ (Fig. 1). Integration of Eq. (1) with respect to y yields

$$\int_0^\delta \frac{\partial}{\partial x}(\rho u r)dy + \rho_e v_e r_e - \rho_w v_w r_w = 0 \tag{2}$$

where $r_e = r_w + \delta \cos\alpha$. The integral term in the foregoing equation can be rewritten as

$$\int_0^\delta \frac{\partial}{\partial x}(\rho u r)dy = \frac{d}{dx}\int_0^\delta \rho u r dy - \rho_e u_e r_e \frac{d\delta}{dx} \tag{3}$$

Equation (2) thus becomes

$$\frac{d}{dx}\int_0^\delta \rho u r dy - \rho_e u_e r_e \frac{d\delta}{dx} + \rho_e v_e r_e - \rho_w v_w r_w = 0 \tag{4}$$

The definition of the displacement thickness δ^* for axisymmetric boundary-layer flow[5] is

$$\int_0^{\delta^*} \rho_e u_e r dy = \int_0^\delta (\rho_e u_e - \rho u) r dy \tag{5}$$

It follows that

$$\int_0^\delta \rho u r dy = \int_{\delta^*}^\delta \rho_e u_e r dy \tag{6}$$

Combining Eqs. (4) and (6) yields

$$\frac{d}{dx}\int_{\delta^*}^\delta \rho_e u_e r dy - \rho_e u_e r_e \frac{d\delta}{dx} + \rho_e v_e r_e - \rho_w v_w r_w = 0 \tag{7}$$

The integral term in Eq. (7) can easily be computed as follows:

$$\int_{\delta^*}^\delta \rho_e u_e (r_w + y)dy = \rho_e u_e r_w(\delta - \delta^*) + \rho_e u_e \frac{1}{2}(\delta^2 - \delta^{*2}) \tag{8}$$

where, for a very slender body of revolution, $\cos\alpha \cong 1$. Combining Eqs. (7) and (8) yields

$$\rho_e v_e r_e = \rho_w v_w r_w + \rho_e u_e r_e \frac{d\delta}{dx} - \frac{d}{dx}\left[\rho_e u_e r_w(\delta - \delta^*) + \rho_e u_e \frac{1}{2}(\delta^2 - \delta^{*2})\right] \tag{9}$$

which can also be written as follows:

$$\frac{v_e}{u_e} = \frac{\rho_w v_w r_w}{\rho_e u_e r_e} + \frac{d\delta}{dx}\left(1 - \frac{r_w}{r_e}\right) + \frac{r_w}{r_e}\left[\frac{d\delta^*}{dx} - \frac{\delta - \delta^*}{\rho_e u_e r_w}\frac{d}{dx}(\rho_e u_e r_w)\right] - \frac{1}{2\rho_e u_e r_e}(\delta^2 - \delta^{*2})\frac{d}{dx}(\rho_e u_e) - \frac{1}{r_e}\left(\delta\frac{d\delta}{dx} - \delta^*\frac{d\delta^*}{dx}\right) \tag{10}$$

The corresponding formula given in Ref. 4 is as follows:

$$\frac{v_e}{u_e} = \frac{\rho_w v_w r_w}{\rho_e u_e r_e} + \frac{d\delta}{dx}\left(1 - \frac{r_w}{r_e}\right) + \left[\frac{d\bar{\delta}^*}{dx} - \frac{\delta - \bar{\delta}^*}{\rho_e u_e r_w}\frac{d}{dx}(\rho_e u_e r_w)\right] \tag{11}$$

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